Generalization of CS Condition

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Introduction

First we recall the definition of CS modules.

Let R be a ring.

- An R-module M is called *CS (extending or C1)* if every closed submodule of M is a direct summand of M.
- R is called *right CS* if R_R is a CS module. Left CS rings can be defined similarly. R is called CS if it is left and right CS.

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Introduction

The CS definition arose not only directly from the study of injectivities but also from work of John von Neumann concerning his attempt to model Quantum Mechanics via continuous geometries.

It seems firstly presented in the following article.

- Y. Utumi, *On continuous regular rings and semisimple self-injective rings*, Canad. J. Math. **12** (1960), 597–605.

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After that, this interesting property of modules has attracted various algebraists to develop relative theory in the next twenty years. The following are two well-known books concentrated on the CS property and related properties of modules.

- S. H. Mohamed, B. J. Müller, *Continuous and Discrete Modules*, Cambridge University Press, 1990.
- N. V. Dung, D. V. Huynh, P. F. Smith, R. Wisbauer, *Extending Modules*, Longman Scientific & Technical, New York, 1994.

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Introduction

Let M be an R-module. Set

- $\mathfrak{C}(M)$=the set of all closed submodules of M,
- $\mathfrak{D}(M)$=the set of all direct summands of M,
- $\mathfrak{S}(M)$=the set of all small submodules of M.

Then M is a CS module means that $\mathfrak{C}(M) = \mathfrak{D}(M)$. This property informs that $\mathfrak{C}(M) \cap \mathfrak{S}(M) = \{0\}$. We consider the property $\mathfrak{C}(M) \cap \mathfrak{S}(M) = \{0\}$, which is equivalent to saying that every nontrivial closed submodule of M is not a small submodule. We call these modules NCS modules. And relative definitions on rings are given. This work is now published in

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Definition 1

Let R be a ring and M be an R-module.

- *M is called an NCS module if every nontrivial closed submodule of M is not a small submodule of M.*
- *R is called a right NCS ring if R_R is an NCS module.*
- *Left NCS rings can be defined similarly.*
- *A ring R is called an NCS ring if it is both left and right NCS.*

Results

Example 2

Every R-module M with $J(M) = 0$ is an NCS module.

It is obvious that every CS module is NCS. The following example shows that the converse is not true.

Example 3

Let k be a division ring and V_k be a right k-vector space of infinite dimension. Take $R = End(V_k)$. Let $S = M_2(R)$. Then S is a left NCS ring but not left CS.

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Results

Lemma 4

[9, Proposition 1.5] Let $A \leq B \leq C$ be a chain of R-modules. If A is closed in B and B is closed in C, then A is closed in C.

Lemma 5

[1, Lemma 5.18] If $K \ll M$ and $f : M \to N$ is a homomorphism then $f(K) \ll N$. In particular, if $K \ll M \leq N$ then $K \ll N$.

Results

Proposition 6

Let R be a ring and M be an R-module. If M is NCS, then every closed submodule of M is also an NCS module.

Corollary 7

Let R be a ring and M be an R-module. If M is NCS, then every direct summand of M is also NCS.

Results

Proposition 8

Let R be a ring and M be an R-module. If every nonzero submodule of M has a uniform submodule, then M is NCS if and only if every uniform closed submodule of M is not a small submodule of M.

Remark 9

There exist modules whose nonzero submodules having no uniform submodules. Let K be any field and $R = \prod_{i=1}^{\infty} K$ be an infinite direct product of copies of K. Set $M = R/\bigoplus_{i=1}^{\infty} K$. Then it is easy to see that M is an R-module whose every nonzero submodule having infinite uniform dimension.

Results

Recall that an R-module M is *finite dimensional* if its Goldie dimension (or uniform dimension) is finite. It is equivalent to saying that M has an essential submodule which is a direct sum of finite uniform submodules of M. If M is finite dimensional or M has an essential socle, it is easy to see that every nonzero submodule of M has a uniform submodule. So we have the following corollary.

Corollary 10

Let R be a ring and M be an R-module. If M is finite dimensional or M has an essential socle, then M is NCS if and only if every uniform closed submodule of M is not a small submodule of M.

Results

Next we will discuss when direct sum of finite NCS modules are still NCS. First we look at when direct sum of two NCS modules are still NCS.

Proposition 11

Let $M = M_1 \oplus M_2$ where M_1 is an NCS module and M_2 is a module with $J(M_2) = 0$. Then M is NCS.

Theorem 12

Let M=$M_1 \oplus M_2$ where M_1 and M_2 are both NCS modules. Then M is NCS if and only if every nonzero closed submodule K of M with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is not a small submodule of M.

Let M and N be two R-modules. Recall that M is called *N-injective* if, for every monomorphism i from K to N and every homomorphism f from K to M, there exists a homomorphism g from N to M such that $f = g \circ i$.

Lemma 13

[5, Lemma 7.5] Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$. Then M_1 is M_2-injective if and only if for every submodule N of M such that $N \cap M_1 = 0$, there exists a submodule M' of M such that $M = M_1 \oplus M'$ and $N \subseteq M'$.

Results

Lemma 14

[11, Proposition 1.5] A module N is $(\oplus_{i \in I} A_i)$-injective if and only if N is A_i-injective for every $i \in I$.

Theorem 15

Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of modules. And for each $i \neq j$, M_i is M_j-injective, $i, j = 1, 2, \ldots, n$. Then M is NCS if and only if each M_i is NCS, $i = 1, 2, \ldots, n$.

Results

Recall that an R-module M is called a *lifting module* if every submodule N of M lies over a direct summand of M. This means that there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2$ is a small submodule of M.

Theorem 16

Let R be a ring and M be an R-module. If M is a lifting and NCS module, then M is a CS module.

Results

By [4, 27.21], If R is a semiperfect ring, then R_R ($_RR$) is a lifting module. So we have the following corollary.

Corollary 17

If R is a semiperfect ring, then R is right NCS if and only if R is right CS.

Results

It is well known that $J(M)$ is the sum of all small submodules of M and the intersection of all maximal submodules of M And if $J(M)$ is a small submodule of M, then $J(M)$ is the largest small submodule containing every small submodule of M.

Proposition 18

Let M be an NCS module with J(M) a small submodule. Then every nontrivial closed submodule of M has a maximal submodule.

Results

It is clear that if M is a finitely generated module or every proper submodule of M is contained in a maximal submodule, then $J(M)$ is a small submodule of M. So we have the following corollary.

Corollary 19

Let M be an NCS module. If M is finitely generated or every proper submodule of M is contained in a maximal submodule, then every nonzero closed submodule of M has a maximal submodule.

Results

Question 20

What the ring R and R-modules will be if every right R-module is NCS?

By [5, 13.3], The following are equivalent.

- Every right R-module is extending.
- Every right R-module is a direct sum of modules of length at most 2.
- Every right R-module N has a decomposition $N = \oplus_{i \in I} N_i$, where N_i is injective and has length 2 or N_i is simple.
- Every cyclic right R-module is a direct sum of injective right R-module and a semisimple module.
- Every cyclic right R-module has finite uniform dimension and the direct sum of (two) uniform modules is extending.

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- Every cyclic right R-module has finite uniform dimension and the direct sum of (two) uniform modules is extending.

Results

Recall that a ring R is called *semiprimitive* if $J(R) = 0$. By Example 2, we have the following example of NCS rings.

Example 21

Every semiprimitive ring is an NCS ring. So every von Neumann regular ring is NCS.

Example 22

A left NCS ring may not be right NCS, even if it is left continuous and two-sided artinian.

Proposition 23

A direct product of rings $R = \prod_{i \in I} R_i$ is right NCS if and only if R_i is right NCS for every $i \in I$.

Results

Lemma 24

Let R be a ring and $e^2 = e \in R$ such that $ReR = R$. If T is a nonzero closed right ideal of eRe, then TR is also a nonzero closed right ideal of R.

Theorem 25

Let R be a ring and $e^2 = e \in R$ such that $ReR{=}R$. If R is right NCS then eRe is also right NCS.

Question 26

Can the condition "$ReR = R$" in the above theorem be removed?

Results

It is natural to ask whether NCS is a Morita invariant of rings or not. The following example shows a matrix ring over a left NCS ring may not be left NCS, even if the ring is left artinian and left CS.

Example 27

(Björk Example) Let F be a field and assume that $a \mapsto \bar{a}$ is an isomorphism $F \to \bar{F} \subseteq F$, where the subfield $\bar{F} \neq F$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. Then R is left NCS, but $M_n(R)$ is not left NCS for each $n \geq 2$.

Results

We use R_n to denote the sets of all $n \times 1$ column matrices over R.

Proposition 28

Let R be a ring and $n \geq 1$. $\mathbb{M}_n(R)$ is right NCS if and only if R_n is NCS as a right R-module.

Using a similar proof as that in Proposition 28, we have the following proposition.

Proposition 29

Let R be a ring and Λ be an infinite set. $\mathbb{CFM}_\Lambda(R)$ is right NCS (CS) if and only if $R_R^{(\Lambda)}$ is NCS (CS).

Results

Proposition 30

Let R be a semiperfect ring and $n \geq 1$. $\mathrm{M}_n(R)$ is right NCS if and only if $\mathrm{M}_n(R)$ is right CS.

Proposition 31

Let R be a right perfect ring and Λ be an infinite set. The following are equivalent.
(1) $R_R^{(\Lambda)}$ *is right NCS;*
(2) $R_R^{(\Lambda)}$ *is right CS;*
(3) $\mathbb{CFM}_\Lambda(R)$ *is right NCS;*
(4) $\mathbb{CFM}_\Lambda(R)$ *is right CS.*

Results

Recall that a ring R is called a *right (countably) Σ-CS ring* if every (countable) direct sum of copies of R_R is a CS module. We call a ring R *right (countably) Σ-NCS ring* if every (countable) direct sum of copies of R_R is an NCS module.

Theorem 32

A ring R is right perfect and right countably Σ-NCS if and only if R is right Σ-CS.

Question 33

Can "right perfect" in the above theorem be replaced by "semiperfect"?

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Thanks